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TWO MEAN VALUES WHICH CHARACTERIZE THE POISSON PROCESS

by

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It is shown that the expected age at time $t > 0$ and the expected residual life at t both single out Poisson processes among renewal processes on the positive line.

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TWO MEAN VALUES WHICH CHARACTERIZE THE POISSON PROCESS

by

Peter Jagers

The literature abounds with characterizations of Poisson processes among renewal processes. Here are two, hopefully, new ones. A suggestion of Kai Lai Chung made me think of them.

Let F be a probability measure on $(0,\infty)$, X_n, $n = 1,2,\ldots$ a sequence of independent random variables with the distribution F, $S_0 = 0$, $S_n = S_{n-1} + X_n$, $n = 1,2,\ldots$ the corresponding partial sums, and

$$N_t = \sup\{n;\ S_n \leq t\}\ , \quad t \geq 0\ ,$$

the induced renewal process. Consider the "age at t",

$$\delta(t) = t - S_{N_t}$$

and the "residual life at t"

$$\delta^*(t) = S_{N_t+1} - t\ , \quad t \geq 0\ .$$

If $\{N_t\}$ is Poisson, that is $F(x) = 1-e^{-\lambda x}$, $x \geq 0$, for some $\lambda > 0$, then, as is well known,

(1) $\delta^*(t)$ has distribution F for all t;

(2) for all t $\delta(t)$ is distributed according to F_t,

$$F_t(x) = \begin{cases} F(x)\ , & 0 \leq x \leq t \\ 1\ , & x > t\ . \end{cases}$$

Chung proved in [1] that either one of (1) or (2) for arbitrary F implies that the process is Poisson. Actually, his results are somewhat stronger, for example, $\delta^*(t)$ must only have the distribution F for a sequence of t:s tending to infinity, or for all t in some initial segment $0 \leq t \leq t_o$, $t_o > 0$. We shall prove the following:

(i) If $E[\delta^*(t)] < \infty$, $t > 0$, is independent of t, then $\{N_t\}$ is Poisson.

(ii) If

$$E[\delta(t)] = \int_o^\infty xF_t(dx) \ , \quad t > 0 \ ,$$

then $\{N_t\}$ is Poisson.

Proof of (i).

$$P\{\delta^*(t) > x\} = \sum_{n=o}^{\infty} P\{\delta^*(t) > x, \ S_n \leq t < S_{n+1}\} =$$

$$= \sum_{n=o}^{\infty} P\{S_{n+1} > t + x, \ S_n \leq t\} =$$

$$= \sum_{n=o}^{\infty} \int_{n=o}^{t} P\{X_{n+1} > t + x - u \mid S_n = u\} F^{*n}(du) =$$

$$= \int_o^t [1 - F(t + x - u)] V(du) \ ,$$

where F^{*n} is the n^{th} convolution power of F and

$$V = \sum_{n=o}^{\infty} F^{*n} \ .$$

Integration yields the expected value

$$\Delta^*(t) = E[\delta^*(t)] = \int_0^\infty P\{\delta^*(t) > x\}dx =$$

$$= \int_0^t \{ \int_{t-u}^\infty [1-F(x)]dx\}V(du)$$

after a change in the order of integration. Since $\Delta^*(t)$ is finite by assumption,

$$\mu = \int_0^\infty xF(dx) < \infty .$$

And if we denote the Laplace-Stieltjes transform by $\hat{}$,

$$\hat{f}(s) = \int_0^\infty e^{-st}f(dt) ,$$

it follows that

$$\hat{\Delta}^*(s) = \{\mu - [1-\hat{F}(s)]s^{-1}\}\hat{V}(s) =$$

$$= [\mu s-1+\hat{F}(s)] \big/ s[1-\hat{F}(s)] , \quad s > 0 .$$

Now if Δ^* is constant, say $\Delta^*(t) = c$, $t \geq 0$ then $\hat{\Delta}^*(s) = c$ for all s and we obtain

$$\hat{F}(s) = [(c-\mu)s+1] \big/ (1+cs) .$$

Since $\hat{F}(\infty) = F(0) = 0$, $c = \mu$ and $\hat{F}(s) = (1+\mu s)^{-1}$, that is $F(x) = 1 - e^{-x/\mu}$.

<u>Proof of (ii)</u>.

$$P\{\delta(t) > x\} = \sum_{n=0}^{\infty} P\{S_n < t-x,\ S_{n+1} > t\} =$$

$$= \sum_{n=0}^{\infty} \int_0^{t-x-} P\{S_{n+1} > t \mid S_n = u\} F*^n(du) =$$

$$= \int_0^{t-x-} [1-F(t-u)]V(du) \ .$$

Therefore

$$\Delta(t) = E[\delta(t)] = \int_0^{\infty} P\{\delta(t) > x\}dx =$$

$$= \int_0^{t} \{ \int_0^{x-} [1-F(t-u)]V(du)\}dx =$$

$$= \int_0^{t-} \{ \int_u^{t} [1-F(t-u)]dx\}V(du) =$$

$$= \int_0^{t-} [1-F(t-u)](t-u)V(du) =$$

$$= \int_0^{t} [1-F(t-u)](t-u)V(du) \ .$$

And the transform is

$$\hat{\Delta}(s) = \hat{V}(s)s \int_0^{\infty} e^{-st}[1-F(t)]tdt =$$

$$= -\hat{V}(s)s \ \frac{d}{ds} \{[1-\hat{F}(s)]s^{-1}\} =$$

$$= [s\hat{F}'(s) + 1 - \hat{F}(s)] / s[1 - \hat{F}(s)] \ .$$

But under (ii)

$$\Delta(t) = \int_0^\infty xF_t(dx) = \int_0^t xF(dx) + t[1-F(t)]$$

yielding

$$\hat{\Delta}(s) = -\hat{F}'(s) + [s\hat{F}'(s) + 1 - \hat{F}(s)]s^{-1} .$$

Equating the two expressions, we see that terms cancel beautifully and

$$s\hat{F}'(s) = \hat{F}^2(s) - \hat{F}(s)$$

with the obvious initial condition

$$\hat{F}(0) = 1 .$$

This is a Riccati equation with the unique solution

$$\hat{F}(s) = (1 + \mu s)^{-1}$$

where

$$\mu = -\hat{F}'(0) = \int_0^\infty xF(dx) < \infty .$$

Hence,

$$F(x) = 1 - e^{-x/\mu} .$$

A simple consequence might be worthwhile noting. Given that $N_t = n$, $t \geq 0$, $n \geq 1$, the random variables $X_1, X_2, \ldots, X_{n-1}$, that is the spans between renewal points in $[0,t]$, have the same distribution. And the last subinterval $t-S_n$ has this same conditional law, for a sequence o. t:s tending to infinity, if and only if the process is Poisson [1].

We get an expectation analogue of this result directly: If, for $t \geq 0$, $n \geq 1$,

$$E[t-S_n|N_t=n] = E[X_1|N_t=n] \ ,$$

then

$$E\delta(t) = \sum_{n=0}^{\infty} E[\delta(t)|N_t = n]P\{N_t = n\} =$$

$$= tP\{N_t = 0\} + \sum_{n=1}^{\infty} E[t-S_n|N_t = n]P\{N_t = n\} =$$

$$= t[1-F(t)] + E[X_1 1_{\{N_t>0\}}] =$$

$$= t[1-F(t)] + \int_0^t xF(dx) = \int_0^{\infty} xF_t(dx) \ .$$

And so, by (ii) it follows that $\{N_t\}$ is Poisson.

Reference

1. Kai Lai Chung, The Poisson process as a renewal process. To appear in Periodica Mathematica.